

Spin characters of generalized symmetric groups

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ABSTRACT. In the context of McKay correspondence and affine Lie algebras, a large part of complex irreducible character values of spin wreath products was computed by vertex operator calculus in [3]. The missing part of the projective character values of the wreath product of the symmetric group and a finite abelian group is determined with the help of Mackey-Wigner method. In particular, this determines projective or spin character values of all classical Weyl groups.

1. Introduction

The spin group \tilde{S}_n is a double cover of the symmetric group S_n . In the seminal paper [15] Schur determined all irreducible projective characters of the symmetric group S_n by introducing a new family of symmetric functions later known as Schur Q -functions. These symmetric functions play the same role for the spin group \tilde{S}_n as Schur functions do for the symmetric group S_n . Schur further showed that the projective character values are on the large part given by Schur Q -functions, but a significant portion was provided by special spin modules and Clifford algebras.

During the last quarter century there has been a resurgence of activities on the spin group and its generalizations. Stembridge [18] gave a combinatorial definition of Schur Q -functions, Sergeev [17] found that the hypercotahedral group of the symmetric group has a similar character theory, Józefiak [8] gave a modern account of Schur's work using superalgebras, Hoffman and Humphreys [4] studied the Hopf algebra structure of the spin characters, Nazarov [12] constructed all irreducible representations of the spin group, and the second author [7] provided a vertex operator approach to Schur Q -functions as well as projective character values. Breakthroughs were also made on modular projective representations of the symmetric groups [2] (see [5] for a survey in this aspect).

On the other hand, recognizing deep connection with McKay correspondence, I. Frenkel, Jing and Wang [3] generalized the first part of Schur's

2000 *Mathematics Subject Classification.* Primary: 20C25; Secondary: 20C30, 20E22.

Key words and phrases. wreath products, spin groups, character table.

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work and determined all irreducible characters of the spin wreath product $\tilde{\Gamma}_n$ of a finite group Γ and the symmetric group S_n . When $\Gamma = 1$, the spin wreath products reduce to \tilde{S}_n . When Γ is a finite cyclic group, they are double covering groups of the generalized symmetric groups, in particular, hyperoctahedral groups when Γ is of order 2. In Schur's original work on \tilde{S}_n , the projective characters of S_n are parameterized by strict partitions. Again in the wreath products, the projective representations are in one to one correspondence to strict partition valued functions or strict colored partitions. In [3] the authors determined all irreducible characters of spin wreath products by vertex operator calculus and also showed that the character values at conjugacy classes of even colored partitions are given by matrix coefficients of products of twisted vertex operators. As in the Schur's case the character values on odd strict colored partitions are beyond the reach of vertex operators. Later in [11] spin characters for the special case of generalized symmetric groups are also determined using group theoretic methods. However the character values on odd strict partition-valued functions are still unknown, as the method associated with McKay correspondence and vertex representations seems not suitable for computing this part of the character table. Knowledge of this part of the character table will be useful in representation theory as they include practically all double coverings of Weyl groups of classical types.

Spin character values have been studied extensively in physics literature as well. In [13] it was observed that plythysms play important role in determining characters for spin characters of $SO(n, \mathbb{C})$ and the spin group \tilde{S}_n . This was later generalized to spin groups associated to orthogonal and symplectic Weyl groups [14] and new algorithms were developed in computing the spin character values of Weyl groups [14, 6].

The purpose of this paper is to obtain the missing part of the character table of spin wreath products $\tilde{\Gamma}_n$ for the cases of abelian group Γ . We construct all irreducible characters by certain induced representations of Young subgroups of $\tilde{\Gamma}_n$ using Mackey-Wigner method of little groups (cf. [16]). Then we can compute all spin character tables of spin wreath products $\tilde{\Gamma}_n$ when Γ is an abelian group. In particular this include, in principle, all irreducible spin character values of Weyl groups of any classical types.

Since abelian groups are direct products of cyclic groups, one only needs to consider the wreath products of cyclic groups and the symmetric groups. In the viewpoint of the new form of McKay correspondence [3] the problem of determination of all spin wreath products of cyclic groups and symmetric groups amounts to realization of twisted affine Lie algebra of type A . On the other hand, Ariki [1] has shown that the Grothendieck group of the category of modules for the cyclotomic Hecke algebra realizes the dual canonical basis for affine Lie algebras of type A , which in turn gives decomposition matrix of the modular representations of the symmetric groups by the Lascoux-Leclerc-Thibon algorithm [9].

The paper is organized as follows. In the first two sections we discuss the basic notions of the wreath products and the Grothendieck group of projective representations of the wreath products. The twisted products of two spin modules are thoroughly reviewed and special attention was paid to the case of cyclic groups. In section three we first recall the basic spin representations and then use the Mackey-Wigner method of little groups to decompose the orbits of Young subgroups. We construct all spin irreducible representations indexed by strict partition-valued functions, and then we show that the character values are sparsely zero and the non-zero values are given according to how the partitions are supported on various conjugacy classes.

2. The spin wreath products $\tilde{\Gamma}_n$.

2.1. The spin group \tilde{S}_n . We first recall some basic properties of the spin group \tilde{S}_n , and then extend them to the spin wreath products $\tilde{\Gamma}_n$. We will not fix Γ to be an abelian group until as late as possible. The spin group \tilde{S}_n is the finite group generated by z and t_i , ($i = 1, \dots, n-1$) with the relations:

$$(2.1) \quad \begin{aligned} z^2 &= 1, \quad t_i^2 = (t_i t_{i+1})^3 = z, \\ t_i t_j &= z t_j t_i, \quad |i - j| > 1, \\ z t_i &= t_i z. \end{aligned}$$

If we replace t_i by $(i, i+1)$ and z by 1, these relations are exactly those for the symmetric group S_n , thus there is a homomorphism θ_n from \tilde{S}_n to S_n sending t_i to the transposition $(i, i+1)$ and z to 1. Clearly the group \tilde{S}_n is a central extension of S_n by the cyclic group \mathbb{Z}_2 . For $n > 3$, Schur [15] has shown that the spin group \tilde{S}_n is one of the two double covers of the symmetric group S_n .

We recall Wales cycle presentation for \tilde{S}_n [19]. For each $k \in \{1, \dots, n\}$, let $x_k = t_k t_{k+1} \cdots t_n \cdots t_{k+1} t_k \in \tilde{S}_{n+1}$. Then for distinct integers $i_1, \dots, i_m \in \{1, 2, \dots, n\}$, we define the cycles in \tilde{S}_n as follows:

$$(2.2) \quad [i_1 i_2 \cdots i_m] = \begin{cases} z, & \text{if } m = 1, \\ x_{i_1} x_{i_m} x_{i_{m-1}} \cdots x_{i_1}, & \text{if } 1 < m \leq n. \end{cases}$$

It is known that $\theta_n([i_1 i_2 \cdots i_m]) = (i_1 i_2 \cdots i_m)$, $\theta_{n+1}(x_i) = (i, n+1)$, and each element of \tilde{S}_n is of the form

$$z^p [i_1 \cdots i_m] [j_1 \cdots j_k] \cdots,$$

where $\{i_1 \cdots i_m\}, \{j_1 \cdots j_k\}, \dots$ is a partition of the set $\{1, 2, \dots, n\}$ and $p \in \mathbb{Z}_2$.

Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition of the positive integer n : $\sum_{i=1}^l \lambda_i = n$, we identify λ with its Ferrers diagram which is formed by the array of n dots having l left-justified rows with row i containing λ_i dots for

$1 \leq i \leq l$. A Young tableau T_λ of shape λ is an assignment of the Ferrers diagram of λ with $1, 2, \dots$. For each tableau T_λ of shape λ with the number $a_{11}, \dots, a_{1\lambda_1}, a_{21}, \dots, a_{2\lambda_2}, a_{l1}, \dots, a_{l\lambda_l}$, we define the element $t_\lambda = [a_{11} \cdots a_{1\lambda_1}][a_{21} \cdots a_{2\lambda_2}] \cdots [a_{l1} \cdots a_{l\lambda_l}]$ of \tilde{S}_n and set $\sigma_\lambda = \theta_n(t_\lambda) = \Pi_{i=1}^l(a_{i1} \cdots a_{i\lambda_i})$ and $t_\lambda^\sigma = \Pi_{i=1}^l[\sigma_\lambda(a_{i1}) \cdots \sigma_\lambda(a_{i\lambda_i})]$.

2.2. The spin group $\tilde{\Gamma}_n$. As we remarked that we will mainly consider the case of $\Gamma = \langle a | a^{r+1} = 1 \rangle$ being a finite cyclic group of order $r+1$. But for convenience we will try to be as general as possible. So we denote by Γ_* the set of conjugacy classes of Γ and $\Gamma^* = \{\gamma_i | i = 0, \dots, r\}$ the set of irreducible characters of Γ with γ_0 being the trivial character.

Let $R(\Gamma) = \bigoplus_{i=0}^r \mathbb{C}\gamma_i$ be the space of complex-valued class functions on Γ . If ζ_c is the order of centralizer of an element in the class $c \in \Gamma_*$, then the order of the class is $|\Gamma|/\zeta_c$. When $\Gamma = \mathbb{Z}_{r+1}$, we have $\zeta_c = r+1$ so each class is of order one.

For a positive integer n , let $\Gamma^n = \Gamma \times \cdots \times \Gamma$ be the n -th direct product of Γ , and let Γ^0 be the trivial group. The action of the spin group \tilde{S}_n on Γ^n is defined as follows:

$$(2.3) \quad \begin{aligned} t_\lambda(g_1, \dots, g_n) &= (g_{\sigma_\lambda^{-1}(1)}, \dots, g_{\sigma_\lambda^{-1}(n)}), \\ z(g_1, \dots, g_n) &= (g_1, \dots, g_n). \end{aligned}$$

The spin wreath product $\tilde{\Gamma}_n = \Gamma \sim \tilde{S}_n$ is the semi-direct product

$$\tilde{\Gamma}_n = \Gamma^n \rtimes \tilde{S}_n = \{(g, t) | g = (g_1, \dots, g_n) \in \Gamma^n, t \in \tilde{S}_n\}$$

with the production

$$(g, t) \cdot (h, s) = (gt(h), ts).$$

Similarly, Γ_n is defined to be the semi-direct product of Γ^n by S_n . It is known that $\tilde{\Gamma}_n$ is a central extension of Γ_n by \mathbb{Z}_2 , thus $|\tilde{\Gamma}_n| = 2n!|\Gamma|^n$.

Let d be the homomorphism from the spin group \tilde{S}_n to group \mathbb{Z}_2 by

$$(2.4) \quad d(t_i) = 1 \ (i = 1, \dots, n-1), \quad d(z) = 0.$$

This homomorphism d is called the *parity* of \tilde{S}_n . Similarly, we define a *parity* for $\tilde{\Gamma}_n$ by

$$(2.5) \quad d(g, t_i) = 1 \ (i = 1, \dots, n-1), \quad d(g, z) = 0.$$

2.3. Partition-valued functions. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ be a partition of n with $\lambda_1 \geq \cdots \geq \lambda_l \geq 1$. We denote by $l = l(\lambda)$ the length of the partition λ and set $|\lambda| = \lambda_1 + \cdots + \lambda_l$. Sometimes we write $\lambda = (1^{m_1} 2^{m_2} 3^{m_3} \dots)$, where m_i is the number of parts in λ which is equal to i .

Given a finite set X , let $\rho = (\rho(x))_{x \in X}$ be a family of partitions indexed by X , we denote by $l(\rho) = \sum_{x \in X} l(\rho(x))$ the length of ρ and by $\|\rho\| = \sum_{x \in X} |\rho(x)|$ the sum of parts of ρ , and then $\rho = (\rho(x))_{x \in X}$ is called a partition-valued function on X . let $\mathcal{P}(X)$ be the set of all partitions indexed by X and $\mathcal{P}_n(X)$ the set of all partitions in $\mathcal{P}(X)$ such that $\|\rho\| = n$. For two partition-valued functions $\rho = (\rho(x))_{x \in X}$ and $\sigma = (\sigma(x))_{x \in X}$, we define

the *union* of $\rho \cup \sigma$ to be the partition-valued function given by $(\rho \cup \sigma)(x) = \rho(x) \cup \sigma(x)$. Here the union of two ordinary partitions is taken to be the juxtaposition of two partitions with their parts rearranged. Subsequently, $\|\rho \cup \sigma\| = \|\rho\| + \|\sigma\|$ and $l(\rho \cup \sigma) = l(\rho) + l(\sigma)$. A partition-valued function is said to be *decomposable* if it is a (non-trivial) union of two or more partition-valued functions.

A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ is called *strict* if $\lambda_i \neq \lambda_j$ for $i \neq j$. We denote by $\mathcal{SP}(X)$ the set of partition-valued functions $(\rho(x))_{x \in X}$ in $\mathcal{P}(X)$ such that each partition $\rho(x)$ is strict. Let $\mathcal{OP}(X)$ be the set of partition-valued functions $(\rho(x))_{x \in X}$ in $\mathcal{P}(X)$ such that all parts of the partitions $\rho(x)$ are odd integers.

For each partition λ we define the *parity* $d(\lambda) = |\lambda| - l(\lambda)$. Similarly, for a partition-valued function $\rho = (\rho(x))_{x \in X}$, we define $d(\rho) = \|\rho\| - l(\rho)$. Then ρ is *even* (or *odd*) if $d(\rho)$ is even (or odd). We let $\mathcal{P}_n^0(X)$ (or $\mathcal{P}_n^1(X)$) to be the collections of even (or odd) partition-valued functions on X .

As convention we denote $\mathcal{SP}_n^i(X) = \mathcal{P}_n^i(X) \cap \mathcal{SP}(X)$ and $\mathcal{OP}_n(X) = \mathcal{P}_n(X) \cap \mathcal{OP}(X)$ for $i \in \{0, 1\}$. For simplicity, $\mathcal{P}(X)$ will be replaced by \mathcal{P} when X consists of a single element. Similarly we have simplified notations such as \mathcal{OP} , \mathcal{SP} , \mathcal{OP}_n and \mathcal{SP}_n^i .

2.4. Split conjugacy classes of $\Gamma^n \rtimes S_\mu$. For $n \geq 1$, and a partition μ of positive integer n , let Ω be the following set

$$(2.6) \quad \{\mu = (\mu_1, \dots, \mu_s) \mid \mu_1 \geq \dots \geq \mu_s \geq 0, |\mu| = n, l(\mu) \leq r + 1\}.$$

For $\mu \in \Omega$, we define the Young subgroup of S_n to be

$$S_{\{1, \dots, \mu_1\}} \times \dots \times S_{\{\mu_1 + \dots + \mu_{s-1} + 1, \dots, \mu_1 + \dots + \mu_s\}},$$

which will be abbreviated as $S_{\mu_1} \times \dots \times S_{\mu_s}$. It is clear that $\Gamma^n \rtimes (S_{\mu_1} \times S_{\mu_2} \times \dots \times S_{\mu_s}) \simeq \Gamma_{\mu_1} \times \dots \times \Gamma_{\mu_s} := \Gamma_\mu$ is a subgroup of Γ_n , also called the Young subgroup of Γ_n .

For $\mu = (\mu_1, \dots, \mu_s) \in \Omega$, let $x = (g, \omega)$ be an element in a conjugacy class of subgroup $\Gamma_{\mu_1} \times \Gamma_{\mu_2} \times \dots \times \Gamma_{\mu_s}$, where $g = (g_1, g_2, \dots, g_n) \in \Gamma$, $\omega = \omega_1 \dots \omega_s \in S_{\mu_1} \times \dots \times S_{\mu_s}$ for $\omega_i \in S_{\mu_i}$. We set $\bar{\rho}^i$ to be the cycle type of ω_i and λ to be the cycle type of ω , thus $\bar{\rho}^i$ is a partition of μ_i and λ is a partition of n . It is known that ω_i can be written as disjoint union of permutations. Each cycle $\sigma_j = (j_1 j_2 \dots j_k)$ in the permutation ω_i corresponds to the element $g_{\sigma_j} = g_{j_k} g_{j_{k-1}} \dots g_{j_1} \in \Gamma$ which is called the cycle-product of σ_j . For each $c \in \Gamma_*$, let $m_k^i(c)$ ($k \geq 0$) be the number of k -cycles in ω_i such that its cycle products lie in the conjugacy class c . let $\rho^i(c) = (1^{m_1^i(c)} 2^{m_2^i(c)} 3^{m_3^i(c)} \dots)$ for each $c \in \Gamma_*$, then $\rho^i = (\rho^i(c))_{c \in \Gamma_*} \in \mathcal{P}_{\mu_i}(\Gamma_*)$. So $\rho = \rho^1 \cup \dots \cup \rho^s \in \mathcal{P}_n(\Gamma_*)$ defines a partition-valued function on Γ_* . In this way we define a bijection ϕ from decomposable partition-valued functions $\rho = \rho^1 \cup \dots \cup \rho^s$ such that $\|\rho^i\| = \mu_i$ to the conjugacy classes of $x = (g, \omega)$ in $\Gamma_{\mu_1} \times \Gamma_{\mu_2} \times \dots \times \Gamma_{\mu_s}$.

Let C_ρ be a conjugacy class in Γ_n of type $\rho = (\rho(c))_{c \in \Gamma_*} \in \mathcal{P}_n(\Gamma_*)$. Fix an order of the conjugacy classes of Γ : c^0, \dots, c^r . Let $T_{\rho(c^i)}$ be the Young

tableau such that the numbers $\sum_{j=1}^{i-1} |\rho(c^j)| + 1, \dots, \sum_{j=1}^i |\rho(c^j)|$ are placed in the squares of the Young diagram of shape $\rho(c^i) = (\rho(c^i)_1, \dots, \rho(c^i)_l)$ from the first row to the last row, and left to right in each row. then we get

$$(2.7) \quad t_{\rho(c^i)} = [a_{i-1} + 1, \dots, a_{i-1} + \rho(c^i)_1] \cdots [a_{i-1} + \rho(c^i)_1 + \dots + \rho(c^i)_{l-1}, \dots, a_{i-1} + |\rho(c^i)|],$$

where $a_{i-1} = \sum_{j=0}^{i-1} |\rho(c^j)|$. Finally, we define $t_\rho = t_{\rho(c^0)} t_{\rho(c^1)} \cdots t_{\rho(c^r)}$ in \tilde{S}_n .

We remark that the general element of $\tilde{\Gamma}_n$ is of the form $(g, z^p t_\rho)$, where ρ is the type of the conjugacy class of $(g, z^p t_\rho)$.

An element $\tilde{x} \in \tilde{\Gamma}_n$ is called *non-split* if \tilde{x} is conjugate to $z\tilde{x}$. Otherwise \tilde{x} is said to be *split*. A conjugacy class of $\tilde{\Gamma}_n$ is called split if its elements are split. An element $x \in \Gamma_n$ is called split if $\theta_n^{-1}(x)$ is split. It is known that the class C_ρ of Γ_n splits if and only if the preimage $\theta_n^{-1}(C_\rho) \triangleq D_\rho$ splits into two conjugacy classes in $\tilde{\Gamma}_n$. For each split conjugacy class C_ρ in Γ_n , we define the conjugacy class D_ρ^+ in $\tilde{\Gamma}_n$ to be the conjugacy class containing the element (g, t_ρ) and define $D_\rho^- = zD_\rho^+$, then $D_\rho = D_\rho^+ \cup D_\rho^-$.

For a partition $\lambda = (1^{m_1} 2^{m_2} 3^{m_3} \dots)$ of n , we denote by $z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!$ the order of the centralizer of an element with cycle type λ in S_n . For each partition-valued function $\rho = (\rho(c))_{c \in \Gamma_*}$, we define

$$(2.8) \quad Z_\rho = \prod_{c \in \Gamma_*} z_{\rho(c)} \zeta_c^{l(\rho(c))}$$

as the order of the centralizer of an element of conjugacy type $\rho = (\rho(c))_{c \in \Gamma_*}$. The order of the centralizer of an element of conjugacy type ρ in $\tilde{\Gamma}_n$ is given by

$$(2.9) \quad \tilde{Z}_\rho = \begin{cases} 2Z_\rho, & C_\rho \text{ is split} \\ Z_\rho, & C_\rho \text{ is non-split} \end{cases}$$

Following the usual definition a representation π of $\tilde{\Gamma}_n$ is called *spin* if $\pi(z) = -1$. In particular, the character values of a spin representation is determined by its values on the split classes, since in that case, for $\tilde{x} \in \tilde{\Gamma}_n$, $\text{trace}(\pi(z\tilde{x})) = -\text{trace}(\pi(\tilde{x})) = 0$ whenever \tilde{x} and $z\tilde{x}$ are conjugate in $\tilde{\Gamma}_n$.

Let $(-1)^d$ be the sign representation of $\tilde{\Gamma}_n$: $\tilde{x} \rightarrow (-1)^{d(\tilde{x})}$. When $(-1)^d \pi \simeq \pi$ we call π a *double spin* representation of $\tilde{\Gamma}_n$. If $\pi' = (-1)^d \pi \not\simeq \pi$, then π' and π are called a pair of *associate spin* representations of $\tilde{\Gamma}_n$.

Let $\rho = (\rho(c))_{c \in \Gamma_*}$ be the type of a conjugacy class C_ρ in Γ_n . It is known that the preimage $\theta_n^{-1}(C_\rho)$ splits into two conjugacy classes in $\tilde{\Gamma}_n$ if and only if $\rho \in \mathcal{OP}_n(\Gamma_*)$ or $\rho \in \mathcal{SP}_n^1(\Gamma_*)$ (see [3]).

3. Twisted Grothendieck groups

In this section we recall some fundamental materials about supermodules, the space of spin super functions and the irreducible spin characters of

\tilde{S}_n , then we study the spin representations of the subgroup $\Gamma^n \rtimes \tilde{S}_\mu$, which is a double cover of Γ_μ .

3.1. Supermodules. Let $\mathbb{C}[\tilde{\Gamma}_n]$ be the group algebra of $\tilde{\Gamma}_n$, thus $\mathcal{A}_n = \mathbb{C}[\tilde{\Gamma}_n]/(1+z)$ has a structure of a \mathbb{Z}_2 -graded algebra by setting $t_i = 1, (i = 1, \dots, n-1)$. It is known that simple superalgebras are of two types:

(1) $M(r|s)$ is a \mathbb{C} -algebra of square $(r+s)$ -matrices with grading determined by the (r,s) -block partition of each matrix: $M(r|s)_0$ consists of matrices with blocks off the main diagonal zero and $M(r|s)_1$ consists of matrices with main diagonal blocks zero.

(2) $Q(n)$ is a subalgebra of $M(n|n)$ consisting of those matrices whose two main diagonal $(n|n)$ -block are equal and two off main diagonal $(n|n)$ -block are equal; the grading is induced from that of $M(n|n)$.

Furthermore, \mathcal{A}_n is semisimple, so it is a direct product of finite number of simple superalgebras. An irreducible spin supermodule (or character) corresponding to $M(r|s)$ will be said to be of type M and the one corresponding to $Q(n)$ of type Q . Any finite dimensional $\mathbb{C}[\Gamma_n]$ -supermodule is isomorphic to a direct sum of simple supermodules of type M and Q .

3.2. The space $R^-(\tilde{\Gamma}_n)$. A spin class function on $\tilde{\Gamma}_n$ is a class function map from $\tilde{\Gamma}_n$ to \mathbb{C} such that $f(zx) = -f(x)$, thus spin class functions vanish on non-split conjugacy classes. A spin super class function on $\tilde{\Gamma}_n$ is a spin class function f on $\tilde{\Gamma}_n$ such that f vanishes further on odd strict conjugacy classes. Let $R^-(\tilde{\Gamma}_n)$ be the \mathbb{C} -span of spin super class functions on $\tilde{\Gamma}_n$.

Let $\tilde{\Gamma}_l \tilde{\times} \tilde{\Gamma}_m$ be the twisted product of $\tilde{\Gamma}_l$ and $\tilde{\Gamma}_m$ with the multiplication

$$(t, t')(s, s') = (tsz^{d(t')d(s)}, t's'),$$

where $s, t \in \tilde{\Gamma}_l, s', t' \in \tilde{\Gamma}_m$ are homogeneous. We define the spin direct product of $\tilde{\Gamma}_l$ and $\tilde{\Gamma}_m$ by

$$(3.1) \quad \tilde{\Gamma}_l \hat{\times} \tilde{\Gamma}_m = \tilde{\Gamma}_l \tilde{\times} \tilde{\Gamma}_m / \{(1, 1), (z, z)\},$$

which can be embedded into the spin group $\tilde{\Gamma}_{l+m}$ canonically by letting

$$(3.2) \quad (t'_i, 1) \mapsto (t_i, 1), \quad (1, t''_j) \mapsto (1, t_{l+j}),$$

where $t'_i \in \tilde{\Gamma}_l (i = 1, \dots, l-1), t''_j \in \tilde{\Gamma}_m (j = 1, \dots, m-1)$. We identify $\tilde{\Gamma}_l \hat{\times} \tilde{\Gamma}_m$ with its image in $\tilde{\Gamma}_{l+m}$ and regard it as a subgroup of $\tilde{\Gamma}_{l+m}$.

In order to study spin modules, we recall the theory of supermodules [8]. The following exposition of supermodules and twisted Grothendieck groups follows [3] closely. For two spin supermodules U and V of $\tilde{\Gamma}_l$ and $\tilde{\Gamma}_m$, we define the super (outer)-tensor product $U \hat{\otimes} V$ by

$$(t, s)(u \hat{\otimes} v) = (-1)^{d(s)d(u)}(tu \hat{\otimes} sv),$$

where s and u are homogeneous elements. Then $U \hat{\otimes} V$ is a spin $\tilde{\Gamma}_l \hat{\times} \tilde{\Gamma}_m$ -supermodule. Moreover, let U and V be an irreducible supermodules for $\tilde{\Gamma}_l$ and $\tilde{\Gamma}_m$ respectively. Then,

- (1) if both U and V are of type M , then $U \hat{\otimes} V$ is a simple $\tilde{\Gamma}_l \hat{\times} \tilde{\Gamma}_m$ -supermodule of type M ;
- (2) if U and V are of different types, then $U \hat{\otimes} V$ is a simple $\tilde{\Gamma}_l \hat{\times} \tilde{\Gamma}_m$ -supermodule of type Q ;
- (3) if both U and V are of type Q , then $U \hat{\otimes} V \simeq N \oplus N$ for some simple $\tilde{\Gamma}_l \hat{\times} \tilde{\Gamma}_m$ -supermodules N of type M .

We consider the twisted Grothendieck group $R^-(\Gamma) = \bigoplus_{n \geq 0} R^-(\tilde{\Gamma}_n)$, and define a multiplication on $R^-(\Gamma)$ as follows. Let $f \in R^-(\tilde{\Gamma}_l)$, $g \in R^-(\tilde{\Gamma}_m)$, and embed $\tilde{\Gamma}_l \hat{\times} \tilde{\Gamma}_m$ in $\tilde{\Gamma}_{l+m}$. Then $f \hat{\times} g$ is an element of $R^-(\tilde{\Gamma}_l \hat{\times} \tilde{\Gamma}_m)$, and we define

$$f \circ g = \text{Ind}_{\tilde{\Gamma}_l \hat{\times} \tilde{\Gamma}_m}^{\tilde{\Gamma}_n} (f \hat{\times} g)$$

which is an element of $R^-(\tilde{\Gamma}_{l+m})$. This gives a bilinear multiplication $R^-(\tilde{\Gamma}_l) \times R^-(\tilde{\Gamma}_m) \rightarrow R^-(\tilde{\Gamma}_{l+m})$, it is known that with this multiplication $R^-(\Gamma)$ is an associative, graded \mathbb{C} -algebra with identity element.

It is known that the irreducible spin super-characters of $\tilde{\Gamma}_n$ form a \mathbb{C} -basis of $R^-(\tilde{\Gamma}_n)$. Let $\phi, \varphi \in R^-(\tilde{\Gamma}_n)$ be two simple supermodules, then the inner product

$$(3.3) \quad \langle \phi, \varphi \rangle = \begin{cases} 1 & \text{if } \phi \simeq \varphi \text{ is type } M, \\ 2 & \text{if } \phi \simeq \varphi \text{ is type } Q, \\ 0 & \text{otherwise.} \end{cases}$$

For a simple supermodule V we define

$$(3.4) \quad \dot{c} = c(V) = \begin{cases} 0 & \text{if } V \text{ is type } M, \\ 1 & \text{if } V \text{ is type } Q, \end{cases}$$

which can be extended to give the type of module V . For simple supermodule $V_1 \hat{\otimes} \cdots \hat{\otimes} V_s$ of \tilde{S}_μ , we can define its type \dot{c} by $c(V_1, V_2) = c(V_1)c(V_2)$.

Let f_i ($i = 1, \dots, s$) be irreducible spin characters of $\tilde{\Gamma}_{\mu_i}$ -supermodule V_i , then $f_1 \hat{\otimes} \cdots \hat{\otimes} f_s$ is an spin super-character of $\tilde{\Gamma}_\mu$. For distinct integers from $\{1, \dots, s\}$, let f_{j_1}, \dots, f_{j_k} be of type Q and $f_{j_{k+1}}, \dots, f_{j_s}$ be of type M , then $f_1 \hat{\otimes} \cdots \hat{\otimes} f_s$ is of type Q or M according to k is odd or even. By the same method as the spin group \tilde{S}_n (cf [7], (6.28)) we can show that

$$(3.5) \quad f_1 \hat{\otimes} \cdots \hat{\otimes} f_s(\tilde{x}_1, \dots, \tilde{x}_s) = (2\sqrt{-1})^{\frac{k-\dot{c}}{2}} f_1(\tilde{x}_1) \cdot f_2(\tilde{x}_2) \cdots f_s(\tilde{x}_s),$$

where \dot{c} is the type $c(V_1 \hat{\otimes} \cdots \hat{\otimes} V_s)$, and the module affords the spin character $f_1 \hat{\otimes} \cdots \hat{\otimes} f_s$.

If $V_1 \hat{\otimes} \cdots \hat{\otimes} V_s$ is of type M , then its underlying $\tilde{\Gamma}_\mu$ -module is also irreducible. If $V_1 \hat{\otimes} \cdots \hat{\otimes} V_s$ is of type Q , then it decomposes itself into two

simple $\tilde{\Gamma}_\mu$ -modules $(V_1 \hat{\otimes} \cdots \hat{\otimes} V_s)^\pm$. Set

$$(3.6) \quad f = \begin{cases} f_1 \hat{\otimes} \cdots \hat{\otimes} f_s & \text{if } f_1 \hat{\otimes} \cdots \hat{\otimes} f_s \text{ is of type } M, \\ (f_1 \hat{\otimes} \cdots \hat{\otimes} f_s)^\pm & \text{if } f_1 \hat{\otimes} \cdots \hat{\otimes} f_s \text{ is of type } Q, \end{cases}$$

Let $f_1 \circ \cdots \circ f_s$ be the induced character of f from $\tilde{\Gamma}_\mu$ to $\tilde{\Gamma}_n$, we shall determine the restriction $Res_{\tilde{\Gamma}_\mu}(f_1 \circ \cdots \circ f_s)$. Let $\{T = \tilde{\Gamma}_\mu t \tilde{\Gamma}_\mu\}$ be the double cosets of $\tilde{\Gamma}_n$ by $\tilde{\Gamma}_\mu, \tilde{\Gamma}_\mu$. For a representative $t \in T$, let $(\tilde{\Gamma}_\mu)_t = t \tilde{\Gamma}_\mu t^{-1} \cap \tilde{\Gamma}_\mu$, a subgroup of $\tilde{\Gamma}_\mu$. By Mackey's decomposition theorem, we have

$$Res_{\tilde{\Gamma}_\mu}(f_1 \circ \cdots \circ f_s) = \bigoplus_{t \in \tilde{\Gamma}_\mu \backslash \tilde{\Gamma}_n / \tilde{\Gamma}_\mu} Ind_{(\tilde{\Gamma}_\mu)_t}^{\tilde{\Gamma}_n}(f^t),$$

where $f^t(\tilde{x}) = f(t^{-1}\tilde{x}t)$. From Frobenius reciprocity we obtain:

$$\begin{aligned} \langle f_1 \circ \cdots \circ f_s, f_1 \circ \cdots \circ f_s \rangle &= \langle f, Res_{\tilde{\Gamma}_\mu}(f_1 \circ \cdots \circ f_s) \rangle \\ &= \sum_{t \in \tilde{\Gamma}_\mu \backslash \tilde{\Gamma}_n / \tilde{\Gamma}_\mu} \langle Res_{(\tilde{\Gamma}_\mu)_t}(f), f^t \rangle_{(\tilde{\Gamma}_\mu)_t}. \end{aligned}$$

When both f and $f_1 \circ \cdots \circ f_s$ are irreducible spin characters, then we have $\langle Res_{(\tilde{\Gamma}_\mu)_t}(f), f^t \rangle_{(\tilde{\Gamma}_\mu)_t} = 0$ for $t \neq 1$. In fact when $t = 1$, one has $\langle Res_{(\tilde{\Gamma}_\mu)_t}(f), f^t \rangle_{(\tilde{\Gamma}_\mu)_t} = \langle f, f \rangle_{\tilde{\Gamma}_\mu} = 1$, and $\langle f_1 \circ \cdots \circ f_s, f_1 \circ \cdots \circ f_s \rangle = 1$. Therefore in this case we have

$$(3.7) \quad \begin{aligned} &\langle f_1 \circ \cdots \circ f_s, f_1 \circ \cdots \circ f_s \rangle_{\tilde{\Gamma}_n} = \langle f, f \rangle_{\tilde{\Gamma}_\mu} \\ &= 2^{k-\dot{c}} \frac{1}{|\tilde{\Gamma}_\mu|} \sum_{\tilde{x} \in \tilde{\Gamma}_\mu} f_1(\tilde{x}_1) \cdots f_s(\tilde{x}_s) \overline{f_1(\tilde{x}_1) \cdots f_s(\tilde{x}_s)}, \end{aligned}$$

where $\tilde{x} = \tilde{x}_1 \cdots \tilde{x}_s$ and $\tilde{x}_i \in \tilde{\Gamma}_{\mu_i} (i = 1, \dots, s)$.

3.3. Irreducible spin representations of \tilde{S}_n . In determining irreducible characters of \tilde{S}_n , Schur introduced the following symmetric function Q_λ .

$$Q_\lambda = Q_{(\lambda_1, \dots, \lambda_l)} = 2^l \sum_{\alpha_1, \dots, \alpha_l=1}^n \frac{x_{\alpha_1}^{\lambda_1} \cdots x_{\alpha_l}^{\lambda_l}}{P_{\alpha_1} \cdots P_{\alpha_l}} A(x_{\alpha_1}, \dots, x_{\alpha_l}),$$

where

$$P_t = \prod_{p=1}^n \frac{x_t - x_p}{x_t + x_p} \quad (t \neq p) \text{ and } A(u_1, u_2, \dots, u_l) = \prod_{1 \leq u_p < u_q \leq n} \left(\frac{u_p - u_q}{u_p + u_q} \right).$$

Schur showed that for $\nu \in \mathcal{SP}_n$, the spin character values $\{\Delta_\nu^\lambda | \lambda \in \mathcal{OP}_n\}$ are determined by

$$(3.8) \quad Q_\nu = \sum_{\lambda \in \mathcal{OP}_n} 2^{\lfloor \frac{l(\nu) + l(\lambda) + d(\nu)}{2} \rfloor} z_\lambda^{-1} \Delta_\nu^\lambda P_\lambda,$$

where $P_\lambda = P_{\lambda_1} P_{\lambda_2} \cdots$ for $\lambda = (\lambda_1, \lambda_2, \dots)$, and $\lceil \frac{l(\nu)+l(\lambda)+d(\nu)}{2} \rceil$ denotes the largest integer $\leq \frac{l(\nu)+l(\lambda)+d(\nu)}{2}$. We recall Schur's result in the following theorem.

THEOREM 3.1. ([15]) *For $n \geq 4$, let $\nu = (\nu_1, \dots, \nu_m) \in \mathcal{SP}_n$ be the irreducible spin character Δ_ν of \tilde{S}_n are determined as follows.*

(i) *If ν is even, there corresponds a unique (double spin) irreducible character Δ_ν whose character values $\{\Delta_\nu^\lambda | \lambda \in \mathcal{OP}_n\}$ are given by (4.8) and $\Delta_\nu^\mu = 0$ for $\mu \notin \mathcal{OP}_n$.*

(ii) *If ν is odd, there are two irreducible (associate spin) characters Δ_ν, Δ'_ν . The character values Δ_ν^λ are given by (4.8) for $\lambda \in \mathcal{OP}_n$, and for other classes they are given by*

$$\Delta_\nu^\nu = (\sqrt{-1})^{(n-l(\nu)+1)/2} \sqrt{\nu_1 \cdots \nu_m / 2},$$

and $\Delta_\nu^\mu = 0$ for $\mu \neq \nu \in \mathcal{SP}_n$. In particular, $(\Delta'_\nu)^\lambda = -\Delta_\nu^\lambda$.

For a combinatorial method to compute the spin characters, see [10].

Let V_{i_j} be a Γ -module affording the character $\gamma_{i_j} \in \Gamma^*$ and W be a spin supermodule of \tilde{S}_μ . For $g = (g_1, \dots, g_n) \in \Gamma^n$, $(g, z^p t_\rho) \in \Gamma^n \rtimes \tilde{S}_\mu$, and distinct integers i_1, \dots, i_s from $\{0, 1, \dots, r\}$, the tensor product $V_{i_1}^{\otimes \mu_1} \otimes \cdots \otimes V_{i_s}^{\otimes \mu_s} \otimes W$ becomes a $\Gamma^n \rtimes \tilde{S}_\mu$ spin supermodule under

$$(3.9) \quad \begin{aligned} & (g, z^p t_\rho) \cdot (v_1 \otimes \cdots \otimes v_n \otimes w) \\ &= (g_1 v_{\sigma_\rho^{-1}(1)} \otimes \cdots \otimes g_n v_{\sigma_\rho^{-1}(n)}) \otimes (z^p t_\rho w), \end{aligned}$$

where $v_1 \otimes \cdots \otimes v_n \in V_{i_1}^{\otimes \mu_1} \otimes \cdots \otimes V_{i_s}^{\otimes \mu_s}$, $w \in W$.

In particular, when $s = 1$ the module $V_i^{\otimes n} \otimes W$ is a spin supermodule of $\tilde{\Gamma}_n$. We remark that the character table of the basic spin supermodules was given in [3].

4. Irreducible spin character tables of $\tilde{\Gamma}_n$.

In this section we first construct irreducible spin supermodules of $\tilde{\Gamma}_n$ induced from Young subgroups $\tilde{\Gamma}_\mu$. By the general theory of spin characters [3] it is enough to focus on strict partition-valued functions of $\tilde{\Gamma}_n$. We will show that only one class of odd strict partition-valued functions can support nonzero irreducible character values.

4.1. The irreducible spin supermodules of $\tilde{\Gamma}_n$. For $j_1, \dots, j_n \in \{0, \dots, r\}$, let $\gamma = \gamma_{j_1} \otimes \gamma_{j_2} \otimes \cdots \otimes \gamma_{j_n}$. Then γ is an irreducible character of Γ^n through the usual tensor product action. Since Γ^n is abelian, they form a group $X = \text{Hom}(\Gamma^n, \mathbb{C}^*)$ under multiplication. The group $\tilde{\Gamma}_n$ acts on X by

$$(\tilde{x} \cdot \gamma)(g) = \gamma(\tilde{x} \cdot g) \quad \text{for } \tilde{x} \in \tilde{\Gamma}_n, \gamma \in X, g \in \Gamma^n.$$

In particular the subgroup \tilde{S}_n acts on X . We will introduce the *class orbit* for the action \tilde{S}_n in X . We denote by I the set of the sequences

$[i_1, i_2, \dots, i_s]$ satisfying $i_a = i_b$ if and only if $a = b$ for $i_a, i_b \in \{0, 1, \dots, r\}$. For a sequence $[i_1, i_2, \dots, i_s] \in I$ and a partition $\mu = (\mu_1, \mu_2, \dots, \mu_s) \in \Omega$, we can get an orbit of \tilde{S}_n in X as follows:

$$(4.1) \quad \begin{aligned} & \mathcal{O}(\gamma_{i_1}^{\otimes \mu_1} \otimes \gamma_{i_2}^{\otimes \mu_2} \otimes \dots \otimes \gamma_{i_s}^{\otimes \mu_s}) \\ &= \{\gamma_{j_1} \otimes \gamma_{j_2} \otimes \dots \otimes \gamma_{j_n} \mid \text{there are } \mu_k \text{ indices equal to } i_k\}, \end{aligned}$$

The set of these orbits are called a *class orbit* with type $\mu = (\mu_1, \dots, \mu_s)$. For simplicity, we denote by Φ_μ the *class orbit* as follows:

$$(4.2) \quad \Phi_\mu \triangleq \{\mathcal{O}(\gamma_{i_1}^{\otimes \mu_1} \otimes \gamma_{i_2}^{\otimes \mu_2} \otimes \dots \otimes \gamma_{i_s}^{\otimes \mu_s}) \mid [i_1, i_2, \dots, i_s] \in I\}.$$

Moreover, we say an irreducible character $\gamma_{j_1} \otimes \gamma_{j_2} \otimes \dots \otimes \gamma_{j_n}$ with type $\mu = (\mu_1, \mu_2, \dots, \mu_s)$ if it is contained in an orbit $\mathcal{O}(\gamma_{i_1}^{\otimes \mu_1} \otimes \gamma_{i_2}^{\otimes \mu_2} \otimes \dots \otimes \gamma_{i_s}^{\otimes \mu_s})$.

LEMMA 4.1. (1) For $\mu \in \Omega$, the number of the class orbits Φ_μ is equal to $|\Omega|$.

(2) For a partition $\mu = (\mu_1, \dots, \mu_s) \in \Omega$, each class orbit Φ_μ contains K_μ orbits, where

$$(4.3) \quad K_\mu = \begin{bmatrix} 1 \\ r+1 \end{bmatrix} \begin{bmatrix} 1 \\ r \end{bmatrix} \dots \begin{bmatrix} 1 \\ r+1-s+1 \end{bmatrix} = \frac{(r+1)!}{(r+1-s)!}.$$

For a sequence $[i_1, i_2, \dots, i_s] \in I$, $\gamma_{i_1}^{\otimes \mu_1} \otimes \gamma_{i_2}^{\otimes \mu_2} \otimes \dots \otimes \gamma_{i_s}^{\otimes \mu_s}$ is a representative of the \tilde{S}_n -orbit $\mathcal{O}(\gamma_{i_1}^{\otimes \mu_1} \otimes \gamma_{i_2}^{\otimes \mu_2} \otimes \dots \otimes \gamma_{i_s}^{\otimes \mu_s})$ in X . For simplicity, we set $\gamma_i^\mu \triangleq \gamma_{i_1}^{\otimes \mu_1} \otimes \gamma_{i_2}^{\otimes \mu_2} \otimes \dots \otimes \gamma_{i_s}^{\otimes \mu_s}$. For a partition $\mu \in \Omega$, let $T_\mu = \{z^p t_\rho \in \tilde{S}_n \mid z^p t_\rho \cdot \gamma_i^\mu = \gamma_i^\mu\}$, then

$$(4.4) \quad T_\mu \simeq \tilde{S}_{\mu_1} \hat{\times} \tilde{S}_{\mu_2} \hat{\times} \dots \hat{\times} \tilde{S}_{\mu_s} = \tilde{S}_\mu.$$

Furthermore, if we set $\tilde{\Gamma}_\mu := \Gamma^n \rtimes T_\mu \simeq \Gamma^n \rtimes \tilde{S}_\mu$, then it can be viewed as a subgroup of $\tilde{\Gamma}_n$.

Next we use the Mackey-Wigner method of little groups (cf. [16]) to construct the irreducible spin characters of $\tilde{\Gamma}_n$. In the following we will let χ_π be an irreducible spin character of \tilde{S}_μ afforded by the spin module π . For abelian groups we may simply use the same letter to denote the representation as well as its character.

Now let $\hat{\pi}$ be the spin module of $\tilde{\Gamma}_\mu$ obtained by composing π with the canonical projection $\tilde{\Gamma}_\mu \rightarrow \tilde{S}_\mu$. Then $\gamma_i^\mu \otimes \hat{\pi}$ is an irreducible spin module of $\tilde{\Gamma}_\mu$. Finally we define

$$\Theta_{\mu,i}^\pi \triangleq \text{Ind}_{\tilde{\Gamma}_\mu}^{\tilde{\Gamma}_n}(\gamma_i^\mu \otimes \chi_{\hat{\pi}}),$$

which is then an irreducible spin character.

4.2. The irreducible spin super character table of $\tilde{\Gamma}_n$. When $n < 4$, the spin group S_n is a direct product of \mathbb{Z}_2 and S_n , so we will assume $n \geq 4$ throughout this section. For $(g, \sigma) \in \Gamma_n$, and σ has type $\rho = (\rho(c))_{c \in \Gamma_*}$. The corresponding elements of $\tilde{\Gamma}_n$ are then $(g, z^p t_\rho)$.

PROPOSITION 4.2. *Let $\mu = (\mu_1, \dots, \mu_s) \in \Omega$ and $\rho = \rho^1 \cup \dots \cup \rho^s \in \mathcal{P}_n(\Gamma_*)$ such that $\rho^j \in \mathcal{P}_{\mu_j}(\Gamma_*)$, then the character values of $\Theta_{\mu,i}^\pi$ at conjugacy classes D_ρ^\pm are given by*

$$(4.5) \quad \Theta_{\mu,i}^\pi(D_\rho^\pm) = \pm K_\rho \prod_{j=1}^s \left(\prod_{c \in \Gamma_*} \gamma_{i_j}^{l(\rho^j(c))} \right) \cdot \chi_\pi(t_\rho),$$

where K_ρ is the number of left cosets T of $\tilde{\Gamma}_\mu$ in $\tilde{\Gamma}_n$ such that $(g, z^p t_\rho)T = T$.

Proof: Since two elements of $\tilde{\Gamma}_n$ are conjugate if and only if they have the same type. So for each transversal t of the left cosets of $\tilde{\Gamma}_\mu$ in $\tilde{\Gamma}_n$, both $(g, z^p t_\rho)$ and $t^{-1}(g, z^p t_\rho)t$ have the same type ρ . Let $V_{i_j} (j = 1, \dots, s)$ be a $\tilde{\Gamma}_n$ -module affording the character $\gamma_{i_j} \in \Gamma^*$. We compute the character $\gamma_i^\mu \otimes \chi_{\hat{\pi}}$ of representation $V_{i_1}^{\otimes \mu_1} \otimes \dots \otimes V_{i_s}^{\otimes \mu_s} \otimes W$. If $\tilde{x} \in \tilde{\Gamma}_m$ and $\tilde{y} \in \tilde{\Gamma}_{n-m}$ then \tilde{x} acts on the first m factors of $V_{i_1}^{\otimes \mu_1} \otimes \dots \otimes V_{i_s}^{\otimes \mu_s}$ and \tilde{y} on the last $n-m$ factors, it is clear that

$$(4.6) \quad \gamma_i^\mu \otimes \chi_{\hat{\pi}}(\tilde{x} \hat{\times} \tilde{y}) = \gamma_i^\mu \otimes \chi_{\hat{\pi}}(\tilde{x}) \cdot \gamma_i^\mu \otimes \chi_{\hat{\pi}}(\tilde{y}).$$

So it is enough to compute $\gamma_i^\mu \otimes \chi_{\hat{\pi}}(g, z^p t_\rho)$ when $g \in \Gamma^n$ and $t_\rho = [1, \dots, \mu_1] \dots [\mu_1 + \dots + \mu_{s-1} + 1, \dots, n]$ is an (μ_1, \dots, μ_s) -cycle. For this purpose, let e_{i_j} be a basis of V_{i_j} (as Γ is a cyclic group) and let $g e_{i_j} = \gamma_{i_j}(g) e_{i_j}$, $\gamma_{i_j}(g) \in \mathbb{C}$. As $t_\rho \cdot \gamma_i^\mu = \gamma_i^\mu$ for $t_\rho \in \tilde{S}_\mu$, it follows that

$$(4.7) \quad \begin{aligned} & (g, z^p t_\rho)(e_{i_1}^{\otimes \mu_1} \otimes \dots \otimes e_{i_s}^{\otimes \mu_s} \otimes w) \\ &= g_1(e_{i_1}) \otimes \dots \otimes g_{\mu_1}(e_{i_1}) \otimes \dots \otimes \\ & \quad g_{\mu_1 + \dots + \mu_{s-1} + 1}(e_{i_s}) \otimes \dots \otimes g_{\mu_1 + \dots + \mu_s}(e_{i_s}) \otimes z^p t_\rho(w) \\ &= \gamma_{i_1}(g_{\mu_1} \dots g_1) \dots \gamma_{i_s}(g_n \dots g_{n-\mu_s})(e_{i_1}^{\otimes \mu_1} \otimes \dots \otimes e_{i_s}^{\otimes \mu_s} \otimes z^p t_\rho(w)) \end{aligned}$$

If for each $j \in \{1, \dots, s\}$, the cycle-product $g_{\Sigma_{k=0}^j \mu_k} \dots g_{\Sigma_{k=0}^{j-1} \mu_k + 2} \cdot g_{\Sigma_{k=0}^{j-1} \mu_k + 1}$ ($\mu_0 = 0$) lies in $c \in \Gamma_*$, then we have

$$(\gamma_i^\mu \otimes \chi_{\hat{\pi}})(g, z^p t_\rho) = \prod_{j=1}^s \left(\prod_{c \in \Gamma_*} \gamma_{i_j}(c)^{l(\rho^j(c))} \right) \chi_\pi(z^p t_\rho).$$

Subsequently

$$(4.8) \quad \begin{aligned} \Theta_{\mu,i}^\pi(D_\rho^\pm) &= \pm \sum_{\tilde{x} \in \tilde{\Gamma}_n} \frac{1}{|\tilde{\Gamma}_\mu|} \gamma_i^\mu \otimes \chi_{\hat{\pi}}(\tilde{x}^{-1}(g, z^p t_\rho)\tilde{x}) \\ &= \pm K_\rho \prod_{j=1}^s \left(\prod_{c \in \Gamma_*} \gamma_{i_j}(c)^{l(\rho^j(c))} \right) \chi_\pi(t_\rho). \quad \square \end{aligned}$$

Throughout this section, set $\Delta_{\bar{\nu}^1} \hat{\otimes} \cdots \hat{\otimes} \Delta_{\bar{\nu}^s}$ to be an irreducible spin super-character of Young subgroup \tilde{S}_μ , where $\Delta_{\bar{\nu}^j}$ is an irreducible spin super-character of \tilde{S}_{μ_j} corresponding to partition $\bar{\nu}^j$. Let χ_π or $(\chi_\pi)^\pm$ be the underlying irreducible spin character according to $\Delta_{\bar{\nu}^1} \hat{\otimes} \cdots \hat{\otimes} \Delta_{\bar{\nu}^s}$ is of type M or type Q .

For $\bar{\nu} = \bar{\nu}^1 \cup \cdots \cup \bar{\nu}^s$, then $\Theta_{\mu,i}^\pi$ is an irreducible spin character of $\tilde{\Gamma}_n$ when $n - l(\bar{\nu})$ is even and it is decomposed into two irreducible spin characters $(\Theta_{\mu,i}^\pi)^\pm$ when $n - l(\bar{\nu})$ is odd.

For $j = 1, \dots, s$, let $\nu^j = (\nu_0^j, \dots, \nu_r^j) = ((\nu_{0,1}^j, \dots, \nu_{0,j_0}^j), \dots, (\nu_{r,1}^j, \dots, \nu_{r,j_r}^j)) \in \mathcal{SP}_{\mu_j}(\Gamma_*)$, then $\nu = \nu^1 \cup \cdots \cup \nu^s$ is in $\mathcal{SP}_n(\Gamma_*)$. For each $c \in \Gamma_*$ and a partition $\lambda \in \mathcal{P}$, we define the characteristic partition-valued function $c^\lambda \in \mathcal{P}(\Gamma_*)$ by

$$c^\lambda(c) = \lambda, \quad c^\lambda(c') = \emptyset, \text{ for } c' \neq c.$$

Let $c^{(\bar{\nu}^j)} := c^{(\nu_{0,1}^j)} \cup \cdots \cup c^{(\nu_{0,j_0}^j)} \cup \cdots \cup c^{(\nu_{r,1}^j)} \cup \cdots \cup c^{(\nu_{r,j_r}^j)}$, clearly this is a special partition-valued function in $\mathcal{SP}_{\mu_j}(\Gamma_*)$ supported only at c . Thus $\tilde{\nu} := c^{(\bar{\nu}^1)} \cup \cdots \cup c^{(\bar{\nu}^s)}$ is a special partition-valued function in $\mathcal{SP}_n(\Gamma_*)$. Let $\bar{\nu} = (\bar{\nu}^1, \dots, \bar{\nu}^s) = \bigcup_{j=1}^s (\bigcup_{c \in \Gamma_*} \nu^j(c))$, where $\bar{\nu}^j = \bigcup_{c \in \Gamma_*} \nu^j(c)$ is a partition of μ_j . For $\nu, \xi \in \mathcal{SP}_n(\Gamma_*)$, we say they are in the same class if $\bar{\nu}$ and $\bar{\xi}$ have the same partition parts, and denote by $\bar{\nu}$ (or $\bar{\xi}$) the type of this class. We denote by $[\tilde{\nu}]$ the set of special partition-valued functions with the type $\bar{\nu}$. It is easy to see that the cardinality of $[\tilde{\nu}]$ is $|\Gamma_*|^{l(\nu)}$. For $(g, \sigma) \in \Gamma_n$, let its corresponding partition-valued function be ν , and denote by $\lambda = (\lambda_1, \dots, \lambda_l)$ the cycle type of σ , then λ is the type of class $[\tilde{\nu}]$.

Let $t_\nu = t_{\nu^1} \cdots t_{\nu^s}$ such that $t_{\nu^i} \in \tilde{S}_{\mu_i}$ for $i \in \{1, \dots, s\}$. If each $\nu^i \in \mathcal{SP}_{\mu_i}^1(\Gamma_*)$ (that is to say $\Delta_{\bar{\nu}^i}$ is of type Q) and s is odd, then we have (cf. (3.5))

$$\begin{aligned} (\chi_\pi)^\pm(t_\nu) &= \pm(2\sqrt{-1})^{\frac{s-1}{2}} \Delta_{\bar{\nu}^1}(t_{\nu^1}) \cdots \Delta_{\bar{\nu}^s}(t_{\nu^s}) \\ (4.9) \quad &= \pm(2\sqrt{-1})^{\frac{s-1}{2}} \cdot (\sqrt{-1})^{\frac{n-l(\nu)+s}{2}} \sqrt{\frac{\lambda_1 \cdots \lambda_l}{2^s}} \\ &= \pm(\sqrt{-1})^{\frac{n-l(\nu)+2s-1}{2}} \sqrt{\frac{\lambda_1 \cdots \lambda_l}{2}}, \end{aligned}$$

where $(\lambda_1, \dots, \lambda_l)$ is the type of ν . Hence

$$(4.10) \quad (\chi_\pi)^\pm(t_\nu) \overline{(\chi_\pi)^\pm(t_\nu)} = \frac{\lambda_1 \cdots \lambda_l}{2},$$

PROPOSITION 4.3. *Let $\mu = (\mu_1, \dots, \mu_s) \in \Omega$ and $\lambda = (\lambda_1, \dots, \lambda_l)$ be the type of $\nu = \nu^1 \cup \cdots \cup \nu^s$. If s is odd and each ν^i in $\mathcal{SP}_{\mu_i}^1(\Gamma_*)$, then $\Theta_{\mu,i}^\pi$ decomposes itself into two irreducible spin characters of $\tilde{\Gamma}_n$. For $\rho \in \mathcal{SP}_n^1(\Gamma_*)$, the character $(\Theta_{\mu,i}^\pi)^\pm$ is given according to*

(i) when $\rho = \rho^1 \cup \dots \cup \rho^s \in [\tilde{\nu}]$, then

$$(\Theta_{\mu,i}^\pi)^\pm(\rho) = \pm K_\rho \prod_{j=1}^s \left(\prod_{c \in \Gamma_*} \gamma_{i_j}(c)^{l(\rho^j(c))} \right) (\sqrt{-1})^{\frac{n-l(\lambda)+2s-1}{2}} \sqrt{\frac{\lambda_1 \cdots \lambda_l}{2}},$$

where K_ρ is the number of left cosets T of $\tilde{\Gamma}_\mu$ in $\tilde{\Gamma}_n$ such that $(g, z^p t_\rho)T = T$.

(ii) when $\rho \notin [\tilde{\nu}]$, one has $(\Theta_{\mu,i}^\pi)^\pm(\rho) = 0$.

Proof: We can get the first assertion from the proposition 4.2 and the Equation (4.9). Now we will prove the second assertion. Since $\Theta_{\mu,i}^\pi$ is an irreducible spin super-character of type Q , there is an pair of underlying associate irreducible spin characters $(\Theta_{\mu,i}^\pi)^\pm$. It is known that the inner product $\langle (\Theta_{\mu,i}^\pi)^\pm, (\Theta_{\mu,i}^\pi)^\pm \rangle$ is equal to 1. Then we have

$$(4.11) \quad 1 = \left(\sum_{\rho \in \mathcal{OP}_n(\Gamma_*)} + \sum_{\rho \in \mathcal{SP}_n^1(\Gamma_*)} \right) \frac{1}{\tilde{Z}_\rho} (\Theta_{\mu,i}^\pi)^\pm(\rho) \overline{(\Theta_{\mu,i}^\pi)^\pm(\rho)}.$$

Moreover, the first part in the above equation equals $\frac{1}{2}$, so the second part also equals $\frac{1}{2}$. By Equation (3.7), we have

$$\begin{aligned} & \sum_{\rho \in \mathcal{SP}_n^1(\Gamma_*)} \frac{1}{\tilde{Z}_\rho} (\Theta_{\mu,i}^\pi)^\pm(\rho) \overline{(\Theta_{\mu,i}^\pi)^\pm(\rho)} \\ &= \sum_{\rho \in \mathcal{SP}_n^1(\Gamma_*)} \frac{1}{\tilde{Z}_\rho} (\gamma_i^\mu \otimes \chi_\pi^\pm)(\rho) \overline{(\gamma_i^\mu \otimes \chi_\pi^\pm)(\rho)} \\ &\geq \sum_{\rho \in [\tilde{\nu}]} \frac{2}{\tilde{Z}_\rho} (\gamma_i^\mu \otimes \chi_\pi^\pm)(D_\rho^+) \overline{(\gamma_i^\mu \otimes \chi_\pi^\pm)(D_\rho^+)} \quad (as \ D_\rho^- = z D_\rho^+) \\ (4.12) \quad &\geq \sum_{\rho \in [\tilde{\nu}]} \frac{1}{\prod_{j=1}^s \left(\prod_{c \in \Gamma_*} z_{\rho^j(c)} \zeta_c^{l(\rho^j(c))} \right)} \chi_\pi^\pm(t_\rho) \overline{\chi_\pi^\pm(t_\rho)} \quad (as \ |\gamma_{i_j}(c)|^2 = 1) \\ &\geq \sum_{\rho \in [\tilde{\nu}]} \frac{1}{\lambda_1 \cdots \lambda_l \cdot (r+1)^{l(\rho)}} \cdot \frac{\lambda_1 \cdots \lambda_l}{2} \quad (by \ (4.10)) \\ &\geq \frac{1}{2}. \end{aligned}$$

Then it follows from (4.11) and (4.12) that

$$1 = \langle (\Theta_{\mu,i}^\pi)^\pm, (\Theta_{\mu,i}^\pi)^\pm \rangle \geq \frac{1}{2} + \frac{1}{2}$$

which forces $(\Theta_{\mu,i}^\pi)^\pm(\rho) = 0$ if $\rho \notin [\tilde{\nu}]$. \square

Example: For $\Gamma = \langle a | a^3 = 1 \rangle$, $n = 13$, set $\Gamma^* = \{\gamma_0, \gamma_1, \gamma_2\}$ and $\Gamma_* = \{c^0, c^1, c^2\}$. Let $\gamma_i(c^j) = w^{ij}$, where $w^3 = 1$ for $w \in \mathbb{C}$.

The first class orbit is $\Phi_{(13)} = \{\mathcal{O}(\gamma_i^{\otimes 13}) = \{\gamma_i^{\otimes 13}\} | i = 0, 1, 2\}$, thus

$$T_{(13)} = \{z^p t_\rho \in \tilde{S}_{13} | z^p t_\rho \cdot \gamma_i^{\otimes 13} = \gamma_i^{\otimes 13}\} \simeq \tilde{S}_{13}, \quad \tilde{\Gamma}_{(13)} = \Gamma^{13} \rtimes T_{(13)} = \tilde{\Gamma}_{13}.$$

Let $i = 1, \nu = ((5, 4, 3, 1))_{c \in \Gamma_*} \in \mathcal{SP}_{13}^1(\Gamma_*)$ and $\rho = ((54)_{c^0}, (31)_{c^2}) \in [\tilde{\nu}]$ (i.e. there are one 5-cycle and one 4-cycle such that their cycle-products lie in c^0 , the same is true for $(31)_{c^2}$), the type of class $[\tilde{\nu}]$ is $\lambda = (5, 4, 3, 1)$. Then

$$(\gamma_1^{\otimes 13} \otimes \Delta_{\bar{\nu}}^{\pm})(D_{\rho}^+) = \pm(\sqrt{-1})^{\frac{13-4+1}{2}} \sqrt{\frac{5 \times 4 \times 3 \times 1}{2}} w = \pm \sqrt{30} w,$$

and $(\gamma_1^{\otimes 13} \otimes \Delta_{\bar{\nu}}^{\pm})(\rho) = 0$ if $\rho \notin [\tilde{\nu}]$. As

$$\begin{aligned} & \sum_{\rho \in [\tilde{\nu}]} \frac{2}{\tilde{Z}_{\rho}} (\gamma_1^{\otimes 13} \otimes \Delta_{\bar{\nu}}^{\pm})(D_{\rho}^+) \overline{(\gamma_1^{\otimes 13} \otimes \Delta_{\bar{\nu}}^{\pm})(D_{\rho}^+)} \\ &= |\Gamma_*|^4 \cdot \frac{1}{5 \cdot 4 \cdot 3 \cdot 1 \cdot 3^4} \cdot |\pm \sqrt{30} w|^2 = \frac{1}{2}. \end{aligned}$$

The second class orbit is $\Phi_{(5,4,4)} = \{\mathcal{O}(\gamma_i^{\otimes 5} \otimes \gamma_j^{\otimes 4} \otimes \gamma_k^{\otimes 4}) \mid i, j, k = 0, 1, 2\}$,

$$T_{(5,4,4)} = \{z^p t_{\rho} \in \tilde{S}_{13} \mid z^p t_{\rho} \cdot \gamma_i^{\otimes 5} \otimes \gamma_j^{\otimes 4} \otimes \gamma_k^{\otimes 4} = \gamma_i^{\otimes 5} \otimes \gamma_j^{\otimes 4} \otimes \gamma_k^{\otimes 4}\} \simeq \tilde{S}_5 \hat{\times} \tilde{S}_4 \hat{\times} \tilde{S}_4,$$

$$\tilde{\Gamma}_{(5,4,4)} = \Gamma^{13} \rtimes (\tilde{S}_5 \hat{\times} \tilde{S}_4 \hat{\times} \tilde{S}_4) = \tilde{\Gamma}_5 \hat{\times} \tilde{\Gamma}_4 \hat{\times} \tilde{\Gamma}_4.$$

For $i = 2, j = 1, k = 0$ and $\bar{\nu} = ((3, 2), (4), (4))$, let χ_{π}^{\pm} be the ordinary spin character of spin super-character $\Delta_{\bar{\nu}^1} \hat{\otimes} \Delta_{\bar{\nu}^2} \hat{\otimes} \Delta_{\bar{\nu}^3}$. For $\rho = (((32)_{c^1}), ((4)_{c^2}), ((4)_{c^1})) \in \mathcal{SP}_{13}^1(\Gamma_*)$, then

$$\gamma_2^{\otimes 5} \otimes \gamma_1^{\otimes 4} \otimes \gamma_0^{\otimes 4} \otimes \chi_{\pi}^{\pm}(\rho) = 2(\sqrt{-1})^{14} \sqrt{\frac{3 \cdot 2 \cdot 4 \cdot 4}{2^3}} = \mp 4\sqrt{3},$$

and $\gamma_2^{\otimes 5} \otimes \gamma_1^{\otimes 4} \otimes \gamma_0^{\otimes 4} \otimes \chi_{\pi}^{\pm} \uparrow_{\tilde{\Gamma}_{(5,4,4)}}^{\tilde{\Gamma}_n}(\rho) = 0$ if $\rho \notin [\tilde{\nu}]$. We see that

$$\begin{aligned} & \sum_{\rho \in [\tilde{\nu}]} \frac{2}{\tilde{Z}_{\rho}} \gamma_2^{\otimes 5} \otimes \gamma_1^{\otimes 4} \otimes \gamma_0^{\otimes 4} \otimes \chi_{\pi}^{\pm}(D_{\rho}^+) \overline{\gamma_2^{\otimes 5} \otimes \gamma_1^{\otimes 4} \otimes \gamma_0^{\otimes 4} \otimes \chi_{\pi}^{\pm}(D_{\rho}^+)} \\ &= |\Gamma_*|^4 \cdot \frac{1}{3 \cdot 2 \cdot 4 \cdot 4 \cdot 3^4} \cdot |\mp 4\sqrt{3}|^2 = \frac{1}{2}, \end{aligned}$$

where the type of class $[\tilde{\nu}]$ is $\lambda = (4, 4, 3, 2)$. \square

For $\nu = \nu^1 \cup \dots \cup \nu^s \in \mathcal{SP}_n(\Gamma_*)$, let $J = \{j_1, \dots, j_k\}$ be a maximal proper subset of $\{1, \dots, s\}$ such that ν^i is in $\mathcal{SP}_{\mu_i}^1(\Gamma_*)$ for $i \in \{j_1, \dots, j_k\}$. Let J' be the complement set of J in $\{1, \dots, s\}$. For $\rho = \rho^1 \cup \dots \cup \rho^s \in \mathcal{SP}_n^1(\Gamma_*)$, one sees that if $\Delta_{\bar{\nu}^i}(t_{\rho^i})$ has nonzero value then ρ^i must be in $[\tilde{\nu}^i]$ for $i \in J$, and ρ^i must be in $\mathcal{OSP}_{\mu_i}(\Gamma_*) := \mathcal{OP}_{\mu_i}(\Gamma_*) \cap \mathcal{SP}_{\mu_i}(\Gamma_*)$ for $i \in J'$. So we have the following results.

THEOREM 4.4. *For $\mu = (\mu_1, \dots, \mu_s) \in \Omega$, let $\nu = \nu^1 \cup \dots \cup \nu^s \in \mathcal{SP}_n^1(\Gamma_*)$. For $\rho \in \mathcal{SP}_n^1(\Gamma_*)$, (i) When $\rho = \rho^1 \cup \dots \cup \rho^s$ satisfying $\rho^i \in [\tilde{\nu}^i]$ for $i \in J$*

and $\rho^i \in \mathcal{OSP}_{\mu_i}(\Gamma_*)$ for $i \in J'$, then

$$(\Theta_{\mu,i}^\pi)^\pm(\rho) = \pm (\sqrt{-1}^{\frac{n-l(\nu)+2k-1}{2}} \sqrt{\frac{\prod_{j \in J} (\prod_{c \in \Gamma_*} \nu^j(c))}{2}}) K_\rho \cdot \prod_{j=1}^s (\prod_{c \in \Gamma_*} \gamma_{i_j}(c)^{l(\rho^j(c))}) \cdot \prod_{j \in J'} \Delta_{\bar{\nu}^j}(t_{\rho^j}),$$

where K_ρ is the number of left cosets T of $\tilde{\Gamma}_\mu$ in $\tilde{\Gamma}_n$ such that $(g, z^p t_\rho)T = T$, and the value of $\prod_{j \in J'} \Delta_{\bar{\nu}^j}(t_{\rho^j})$ is determined by wreath products of Schur Q -functions (see [3]).

(ii) $(\Theta_{\mu,i}^\pi)^\pm(\rho) = 0$, otherwise.

Proof: The first assertion follows from the Equations (4.8), (4.9) and Proposition 4.4.

Now we consider the second part. For $\nu = \nu^1 \cup \dots \cup \nu^s \in \mathcal{SP}_n^1(\Gamma_*)$, we suppose that there are $2k-1$ irreducible spin super-characters of type Q and $s-2k+1$ of type M in $\{\Delta_{\bar{\nu}^1}, \dots, \Delta_{\bar{\nu}^s}\}$. Moreover, we can assume that $\Delta_{\bar{\nu}^1}, \dots, \Delta_{\bar{\nu}^{2k-1}}$ are of type Q and $\Delta_{\bar{\nu}^{2k}}, \dots, \Delta_{\bar{\nu}^s}$ are of type M . Then by the Equation (4.9) and the second Equation of (4.12), we have

$$\begin{aligned} & \sum_{\rho \in \mathcal{SP}_n^1(\Gamma_*)} \frac{1}{\tilde{Z}_\rho} (\Theta_{\mu,i}^\pi)^\pm(\rho) \overline{(\Theta_{\mu,i}^\pi)^\pm(\rho)} \\ &= \sum_{\rho \in \mathcal{SP}_n^1(\Gamma_*)} \frac{2^{2(k-1)}}{\tilde{Z}_\rho} \left| \prod_{j=1}^s (\prod_{c \in \Gamma_*} \gamma_{i_j}(c)^{l(\rho^j(c))}) \Delta_{\bar{\nu}^j}(\rho^j) \right|^2 \\ (4.13) \quad &= \sum_{\rho = \rho^1 \cup \dots \cup \rho^s \in \mathcal{SP}_n^1(\Gamma_*)} \frac{2^{2(k-1)}}{\tilde{Z}_\rho} \left| \prod_{j=1}^s \Delta_{\bar{\nu}^j}(\rho^j) \right|^2 \quad (as \quad |\gamma_{i_j}(c)|^2 = 1) \\ &\geq \sum_{\rho^j \in [\tilde{\nu}^j]: j \in J; \rho^j \in \mathcal{SP}_{\mu_j}(\Gamma_*): j \in J'} 2^{2k-3} \left(\prod_{j=1}^s \frac{1}{\tilde{Z}_{\rho^j}} |\Delta_{\bar{\nu}^j}(\rho^j)|^2 \right) \end{aligned}$$

For each $j \in J$, $\frac{1}{\tilde{Z}_{\rho^j}} |\Delta_{\bar{\nu}^j}(\rho^j)|^2$ is a constant for different $\rho^j \in [\tilde{\nu}^j]$ because of they have the same type $\bar{\nu}^j$. Hence $\prod_{j=1}^{2k-1} \frac{1}{\tilde{Z}_{\rho^j}} |\Delta_{\bar{\nu}^j}(\rho^j)|^2$ is a constant for different $\rho^1 \cup \dots \cup \rho^{2k-1} \in \mathcal{SP}_{\mu_1}(\Gamma_*) \cup \dots \cup \mathcal{SP}_{\mu_{2k-1}}(\Gamma_*)$, then the above Equation (4.13)

$$(4.14) \quad \geq 2^{2k-3} \left(\sum_{\rho^j \in [\tilde{\nu}^j]: j \in J} \prod_{j=1}^{2k-1} \frac{|\Delta_{\bar{\nu}^j}(\rho^j)|^2}{\tilde{Z}_{\rho^j}} \right) \left(\sum_{\rho^j \in \mathcal{OSP}_{\mu_j}(\Gamma_*): j \in J'} \prod_{j=2k}^s \frac{|\Delta_{\bar{\nu}^j}(\rho^j)|^2}{\tilde{Z}_{\rho^j}} \right)$$

In the above equation, $\mathcal{OSP}_{\mu_j}(\Gamma_*) := \mathcal{OP}_{\mu_j}(\Gamma_*) \cap \mathcal{SP}_{\mu_j}(\Gamma_*)$, and

$$\begin{aligned}
 & \sum_{\rho^j \in \mathcal{OSP}_{\mu_j}(\Gamma_*): j \in J'} \prod_{j=2k}^s \frac{|\Delta_{\bar{\nu}^j}(\rho^j)|^2}{Z_{\rho^j}} \\
 &= \sum_{\rho^j \in \mathcal{OSP}_{\mu_j}(\Gamma_*): j \in J'} \prod_{j=2k}^s \frac{|\Delta_{\bar{\nu}^j}(D_{\rho^j})|^2}{\prod_{c \in \Gamma_*} z_{\rho^j(c)} \zeta_c^{l(\rho^j(c))}} \\
 &= \sum_{\bar{\rho}^j \in \mathcal{OP}_{\mu_j}} \prod_{j=2k}^s \frac{|\Gamma_*|^{l(\rho^j)} |\Delta_{\bar{\nu}^j}(t_{\rho^j})|^2}{z_{\bar{\rho}^j}(r+1)^{l(\rho^j)}} \\
 &= \prod_{j=2k}^s \left(\sum_{\bar{\rho}^j \in \mathcal{OP}_{\mu_j}} \frac{|\Delta_{\bar{\nu}^j}(t_{\rho^j})|^2}{z_{\bar{\rho}^j}} \right) \\
 &= \prod_{j=2k}^s \langle \Delta_{\bar{\nu}^j}, \Delta_{\bar{\nu}^j} \rangle_{\tilde{S}_{\mu_j}} \\
 &= 1.
 \end{aligned} \tag{4.15}$$

Subsequently, the Equation (4.13)

$$\begin{aligned}
 & \geq 2^{2k-3} \left(\sum_{\rho^j \in [\bar{\nu}^j]: j \in J} \prod_{j=1}^{2k-1} \frac{|\Delta_{\bar{\nu}^j}(\rho^j)|^2}{Z_{\rho^j}} \right) \\
 & \geq \sum_{\rho^j \in [\bar{\nu}^j]: j \in J} \frac{2^{2(k-1)} \prod_{j=1}^{2k-1} |(\Delta_{\bar{\nu}^j})^\pm(D_{\rho^j}^+)|^2}{\prod_{j=1}^{2k-1} \left(\prod_{c \in \Gamma_*} z_{\rho^j(c)} \zeta_c^{l(\rho^j(c))} \right)} \quad (D_\rho^- = z D_\rho^+) \\
 & \geq 2^{2(k-1)} \prod_{j=1}^{2k-1} \frac{|\Gamma_*|^{l(\rho^j)} \left| \sqrt{\prod_{c \in \Gamma_*} \nu^j(c)/2} \right|^2}{\prod_{c \in \Gamma_*} \nu^j(c) (r+1)^{l(\rho^j(c))}} \\
 & \geq \frac{1}{2}
 \end{aligned} \tag{4.16}$$

It is known that $\sum_{\rho \in \mathcal{OP}_n(\Gamma_*)} \frac{1}{Z_\rho} \Theta_{\mu,i}^\pi(\rho) \overline{\Theta_{\mu,i}^\pi(\rho)} = 1/2$, which forces $\Theta_{\mu,i}^\pi(\rho) = 0$ if $\rho^i \notin [\bar{\nu}^i]$ for $i \in J$ or $\rho^i \notin \mathcal{OSP}_{\mu_i}(\Gamma_*)$ for $i \in J'$. \square

COROLLARY 4.5. *For $\mu = (\mu_1, \dots, \mu_s) \in \Omega$, let $\nu = \nu^1 \cup \dots \cup \nu^s$ be in $\mathcal{SP}_n^0(\Gamma_*)$.*

(i) When $\rho = \rho^1 \cup \dots \cup \rho^s \in \mathcal{OP}_n(\Gamma_)$ and $\|\rho^i\| = \mu_i$, the value of $\Theta_{\mu,i}^\pi(D_\rho^\pm)$ is determined by wreath products of Schur Q -functions (see [3]).*

(ii) Otherwise, one has $\Theta_{\mu,i}^\pi(D_\rho^\pm) = 0$.

Proof: (i) If $\rho^i \in \mathcal{OP}_{\mu_i}(\Gamma_*)$, the value of each $\Delta_{\bar{\nu}^i}(t_{\rho^i})$ is determined by Schur Q-functions given by (4.8), then the value of $\Theta_{\mu,i}^\pi(D_\rho^\pm)$ is determined by wreath products of Schur Q-functions.

(ii) If ρ can not be decomposed as $\rho^1 \cup \cdots \cup \rho^s$ such that $\|\rho^i\| = \mu_i$, then it is easy to see that $\Theta_{\mu,i}^\pi(D_\rho^\pm) = 0$. As $\nu \in \mathcal{SP}_n^0(\Gamma_*)$, we can suppose that there are $2k$ irreducible spin characters of type Q and $s - 2k$ of type M in $\{\Delta_{\bar{\nu}^1}, \dots, \Delta_{\bar{\nu}^s}\}$. Similarly, we can assume that $\Delta_{\bar{\nu}^1}, \dots, \Delta_{\bar{\nu}^{2k}}$ are of type Q and $\Delta_{\bar{\nu}^{2k+1}}, \dots, \Delta_{\bar{\nu}^s}$ are of type M . Then by the Equation (4.13), we see that $\sum_{i=1}^{2k} d(\rho^i) = 2k$. If $\rho = \rho^1 \cup \cdots \cup \rho^s \in \mathcal{SP}_n^1(\Gamma_*)$, then at least one ρ^i is not in $\mathcal{OS}\mathcal{P}_{\mu_i}(\Gamma_*)$ for $i = 2k + 1, \dots, s$, so $\Delta_{\bar{\nu}^i}(t_{\rho^i}) = 0$. Therefore we must have $\Theta_{\mu,i}^\pi(\rho) = 0$. \square

COROLLARY 4.6. *For $j \in \{1, \dots, s\}$, let $\Delta_{\bar{\nu}^j}$ be the basic spin super-character of \tilde{S}_{μ_j} , then $\Theta_{\mu,i}^\pi$ is the basic spin super-character of $\tilde{\Gamma}_n$. If each μ_j is even and s is odd, then $\Theta_{\mu,i}^\pi$ is of type Q and the values of the basic spin characters $(\Theta_{\mu,i}^\pi)^\pm$ at the conjugacy class D_ρ^+ of type $\rho = \rho^1 \cup \cdots \cup \rho^s \in \mathcal{SP}_n^1(\Gamma_*)$ are*

$$(\Theta_{\mu,i}^\pi)^\pm(D_\rho^+) = \begin{cases} \pm(\sqrt{-1})^{\frac{n+s-1}{2}} \sqrt{\frac{\mu_1 \cdots \mu_s}{2}} \prod_{k=1}^s \gamma_{i_k}(c^{j_k}), & \rho^i = c_{j_i}^{(\mu_i)}, \\ 0, & \text{otherwise.} \end{cases}$$

Acknowledgments

The second author gratefully acknowledges the partial support of Max-Planck Institut für Mathematik in Bonn, Simons Foundation grant 198129, NSFC grant 10728102, and NSF grants 1014554 and 1137837 during this work.

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